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# Convex Optimization for Dynamical Systems
# Region of Attraction Computation via Polynomial Optimization

Region of Attraction (ROA) estimation problems yield interesting optimization approaches when considering polynomial dynamics. In this study, a method presented in [2] is shown, providing an outer approximation to the ROA, solving a Polynomial Optimization problem, induced by occupation measures, approximating an initial infinite dimensional problem with a hierarchy of finite dimensional linear matrix inequalities, with convergence for an arbitrarily large polynomial degree.

## I. Introduction

The problem relies on computing the Region of Attraction (ROA) of a target set, on a nonlinear dynamical system. The domain sets are $X, U, X_T$, respectively for the trajectories, control, and terminal state.

$$\dot{x}(t) = f(t, x(t), u(t)) \quad t \in [0, T] \tag{I.1}$$

where it is considered that each entry is polynomial, i.e., $f_i \in \mathbb{R}[t, x, u], i \in \mathbb{Z}_{[1,n]}$.
The set of all admissible trajectories starting from $x_0$ is given by:

$$\mathcal{X}(x_0) := \{x(.) : \exists u(.) \in \mathcal{U} \text{ s.t. } \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e., } x(0) = x_0, x(t) \in X, x(T) \in X_T, \forall t \in [0, T]\} \tag{I.2}$$

where $\mathcal{U}$ denotes the set of all control functions, and $x(.)$ is absolutely continuous. Also, a.e. stands for "almost everywhere" in a Lebesgue measure or a weak sense.
The defined sets $X, U, X_T$ form semialgebraic constraints that restrict state and control input.

$$x(t) \in X := \{x \in \mathbb{R}^n : g_i^X(x) \geq 0, i \in \mathbb{Z}_{[1,n_X]}\}, \quad t \in [0, T] \tag{I.3}$$
$$u(t) \in U := \{u \in \mathbb{R}^m : g_i^U(u) \geq 0, i \in \mathbb{Z}_{[1,n_U]}\}, \quad t \in [0, T] \tag{I.4}$$

where $g_i^X \in \mathbb{R}[x], g_i^U \in \mathbb{R}[x]$, and the terminal state $x(T)$ is constrained to lie in the basic semialgebraic set

$$X_T := \{x \in \mathbb{R}^n : g_i^{X_T}(x) \geq 0, i \in \mathbb{Z}_{[1,n_T]}\} \subset X, \quad g_i^{X_T} \in \mathbb{R}[x] \tag{I.5}$$

**Definition I.1** (Region of Attraction (ROA) [2])**.** The Region of Attraction (ROA) of a dynamical system is the set of all initial states that can be directed to a target set, with an admissible control, while remaining in the state-constraint set. The ROA is denoted by

$$X_0 = \{x_0 \in X : \mathcal{X}(x_0) \neq \emptyset\} \tag{I.6}$$

### A. Assumptions

1. **Assumption 1: $X$, $U$, and $X_T$ are compact**. Therefore, the sets are also bounded.
2. Assumption 2: The control system satisfies $\lambda(X_0) = \lambda(\bar{X}_0)$. Thus, the volume of the relaxed region has the same volume as the ROA measure.
3. Assumption 3: Archimedian semialgebraic sets. Allows us to use quadratic modules (weighted SOS).

**Definition I.2** (Quadratic Module [4])**.** The quadratic module generated by the polynomials $g_1, \ldots, g_m$ is the set

$$Q(g_1, \ldots, g_m) := \left\{ \sigma_0 + \sum_{i=1}^{m} g_i \sigma_i : \sigma_0, \ldots, \sigma_m \in \Sigma[x] \right\} \tag{I.7}$$

**Definition I.3** (Archimedian semialgebraic sets)**.** The Quadratic module $Q(g_1, \dots, g_m)$ is said to be Archimedean if $\exists N \in \mathbb{N} : N - (x_1^2 + \dots + x_n^2) \in Q(g_1, \dots, g_m)$.

Assumptions 1 and 3 are sufficient conditions to apply Putinar's theorem [5], and then a Quadratic module can be a representation of the semialgebraic sets.

**Theorem I.4** (Putinar [5])**.** *Suppose the polynomials $g_1, \dots, g_m$ are such that $Q(g_1, \dots, g_m)$ is Archimedean. If $f \in \mathbb{R}[x]$ is positive on the set $K$, then $f \in Q(g_1, \dots, g_m)$.*

## II. Occupation measures

[2] shows that it is possible to compute the ROA by solving a convex linear program (LP), with infinite dimensionality. However, the infinite dimensional problem can be converted to an equivalent optimization problem, with finite dimensionality, which is more suitable for numerical simulation.

**Definition II.1** (Occupation measure)**.** Given an initial condition $x_0$ and admissible $x(.|x_0) \in \mathcal{X}_0$, with $u(.|x_0) \in \mathcal{U}$, the occupation measure $\mu(.|x_0)$ is defined by

$$\mu(A \times B \times C|x_0) = \int_0^T \mathbb{I}_{A\times B\times C}(t, x(t|x_0), u(t|x_0))dt, \quad \forall A \times B \times C \subset [0, T] \times X \times U \tag{II.1}$$

where $\mu(A \times B \times C|x_0)$ quantifies the amount of time of $A \subset [0, T]$ spent with the trajectory state and control $(x.|x_0), u(.|x_0)) \in B \times C \subset X \times U$.

For any measurable function $g(t, x, u)$ we have the property

$$\int_0^T g(t, x(t|x_0), u(t|x_0))dt = \int_{[0,T]\times X\times U} g(t, x, u)d\mu(t, x, u|x_0) \tag{II.2}$$

which relates the measure $\mu(.|x_0)$ to the trajectory state and control in the given time.
Furthermore, a particularly relevant relation between a measure $\mu$ and the function $f$ is the Liouville equation, with the derivative of the measure in a weak or distributional sense:

$$\partial_t\mu + \mathrm{div}_x(f\mu) = 0, \quad \mu \in M([0, T] \times X) \tag{II.3}$$

And with a test function, initially with flexible regularity $C_0^\infty([0, T] \times X)$, we integrate with time and space. By integrating by parts and shifting the derivative to the test function with the first term, and applying the divergence theorem with the second term:

$$\int_0^T \int_X (\partial_t\mu + \mathrm{div}_x(f\mu))vdxdt = \int_0^T \int_X \partial_t\mu \cdot vdxdt + \int_0^T \int_X \mathrm{div}\ (f\mu) \cdot vdxdt \tag{II.4}$$

$$= -\int_0^T \int_X \mu \cdot \partial_t vdxdt - \int_0^T \int_X f\mu \cdot \nabla vdxdt = 0 \tag{II.5}$$

$$\int_0^T \int_X (\partial_t v + \nabla v \cdot f)\mu dxdt = 0 \tag{II.6}$$

$$\Rightarrow \int_0^T \int_X (\partial_t v + \nabla v \cdot f)\, d\mu(t, x) = 0, \quad \forall v \in C_0^\infty([0, T] \times X) \tag{II.7}$$

We then reformulate II.3 via the Linear Operator $\mathcal{L} : C^1([0, T] \times X) \to C([0, T] \times X \times U)$ with

$$v \mapsto Lv := \frac{\partial v}{\partial t} + \nabla v \cdot f \tag{II.8}$$

which has an adjoint $\mathcal{L}' : C([0, T] \times X \times U)' \to C_1([0, T] \times X)'$, with

$$\begin{gathered}\langle \mathcal{L}'\nu, v\rangle := \langle \nu, \mathcal{L}v\rangle = \int_{[0,T]\times X\times U} \mathcal{L}v(t, x, u)d\nu(t, x, u) \\ \forall \nu \in M([0, T] \times X \times U) = C([0, T] \times X \times U)', v \in C^1([0, T] \times X)\end{gathered} \tag{II.9}$$

Often the adjoint is expressed with $\mathcal{L}'\nu = -\partial_t\nu - \nabla \cdot f\nu$. Note that the liouville equation II.3 is denoted with the given adjoint.

Given a test function $v \in C^1([0,T] \times X)$,

$$\begin{aligned} v(T, x(T|x_0)) = v(0,x_0) + \int_0^T \frac{d}{dt} v(t, x(t|x_0))dt = v(0,x_0) + \int_0^T \mathcal{L}v(t, x(t|x_0), u(t|x_0))dt \\ = v(0,x_0) + \int_{[0,T]\times X \times U} \mathcal{L}v(t,x,u)d\mu(t,x,u|x_0) = v(0,x_0) + \langle \mathcal{L}'\mu(.|x_0), v\rangle \\ \Rightarrow v(T, x(T|x_0)) = v(0,x_0) + \langle \mathcal{L}'\mu(.|x_0), v\rangle \end{aligned} \tag{II.10}$$

The average occupation measure $\mu$ quantifies the average time spent with the given control and state.
By integrating II.10 w.r.t $\mu_0$, we obtain

$$\int_{X_T} v(T, x(T|x_0))d\mu_T(x) = \int_X v(0,x_0)d\mu_0(x) + \int_{[0,T]\times X\times U} \mathcal{L}v(t,x,u)d\mu(t,x,u) \tag{II.11}$$

$$\langle \mu_T, v(T,.)\rangle = \langle \mu_0, v(0,.)\rangle + \langle \mu, \mathcal{L}v\rangle \tag{II.12}$$

which relate the nonnegative measures $\mu \geq 0, \mu_0 \geq 0, \mu_T \geq 0$.
We can rewrite II.12 with the Dirac $\delta_t$ measure as $\langle \mu_0, v(0,.)\rangle = \langle \delta_0 \otimes \mu_0, v\rangle$, obtaining

$$\langle \delta_T \otimes \mu_T, v\rangle = \langle \delta_0 \otimes \mu_0, v\rangle + \underbrace{\langle \mu, \mathcal{L}v\rangle}_{\langle \mathcal{L}'\mu, v\rangle} \quad \forall v \in C^1([0,T]\times X) \tag{II.13}$$

And finally, we obtain the linear operator equation:

$$\delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 + \mathcal{L}'\mu \tag{II.14}$$

Relaxation can be obtained by assuming suitable changes:

$$\rightarrow \dot{x}(t) \in f(t, x(t), U) \tag{II.15}$$
$$\rightarrow \dot{x}(t) \in \text{conv} f(t, x(t), U) \tag{II.16}$$

which can be shown to satisfy Liouville's equation II.14.
Then, the set of admissible trajectories and the ROA are defined with

$$\begin{aligned} \bar{\mathcal{X}}(x_0) := \{x(.) : \exists u(.) \in U \text{ s.t. } \dot{x}(t) \in \text{conv} f(t, x(t), U) \text{ a.e.}, \\ x(0) = x_0 \in x(t) \in X, x(T) \in X_T, \forall t \in [0,T]\} \end{aligned} \tag{II.17}$$

$$\bar{X}_0 = \{x_0 \in X : \bar{\mathcal{X}}(x_0) \neq \emptyset\} \tag{II.18}$$

## III. ROA via Optimization

Computing the ROA then can be determined with the optimization problem below

$$\begin{aligned} q^* = \sup \quad & \lambda(\text{spt } \mu_0) \\ \text{s.t.} \quad & \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 + \mathcal{L}'\mu \\ & \mu \geq 0, \mu_0 \geq 0, \mu_T \geq 0 \\ & \text{spt } \mu \subset [0,T] \times X \times U \\ & \text{spt } \mu_0 \subset X, \text{spt } \mu_T \subset X_T \end{aligned} \tag{III.1}$$

**Lemma III.1** (ROA computation via III.1)**.** *It can be shown that the volume in III.1 equals to the volume of the ROA, i.e.,* $q^* = \lambda(X_0)$.

*Proof.* By definition, for $x_0 \in X_0$, there is an admissible trajectory in $\mathcal{X}(x_0)$, thus, for $\mu_0$ with spt $\mu_0 \subset X_0$, there exist $\mu, \mu_T$ satisfying the constraints.
With Assumption 2, $\lambda(X_0) = \lambda(\bar{X}_0)$, then $q^* \geq \lambda(X_0) = \lambda(\bar{X}_0)$.
Supposing by contradition $(\mu_0, \mu, \mu_T)$ satisfy the constraints and $\lambda(\text{spt } \mu_0 \setminus \bar{X}_0) > 0$. However, no trajectory from spt $\mu_0 \setminus \bar{X}_0$ can be admissible, then $\lambda(\text{spt } \mu_0 \setminus \bar{X}_0) = 0 \Rightarrow \lambda(\text{spt } \mu_0) \leq \lambda(\bar{X}_0)$. Then, $q^* \leq \lambda(\bar{X}_0) = \lambda(X_0)$. □

It is possible to reformulate III.1, by using that for any feasible $\mu_0$:

$$\lambda(\text{spt } \mu_0) \leq \lambda(X_0) \tag{III.2}$$
$$\mu_0 \leq \lambda \Rightarrow \mu_0(X) = \mu_0(\text{spt } \mu_0) \leq \lambda(\text{spt } \mu_0) \leq \lambda(X_0) \tag{III.3}$$
$$\therefore \mu_0(X) \leq \lambda(X_0) \tag{III.4}$$

Also, by definition $\mu_0(X) \geq \lambda(X_0)$. Thus: $p^* := \mu_0(X) = \lambda(X_0)$, and we obtain:

$$\begin{aligned} p^* = \sup \quad & \mu_0(X) \\ \text{s.t.} \quad & \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 + \mathcal{L}'\mu \\ & \mu \geq 0, \quad \lambda \geq \mu_0 \geq 0, \quad \mu_T \geq 0 \\ & \text{spt } \mu \subset [0,T] \times X \times U \\ & \text{spt } \mu_0 \subset X, \text{spt } \mu_T \subset X_T \end{aligned} \tag{III.5}$$

**Theorem III.2.** *The optimal value of the infinite-dimensional problem is equal to the volume of the ROA, i.e.,*

$$p^* = \lambda(X_0) \tag{III.6}$$

*Proof.* From Lemma III.1 $\lambda(\text{spt } \mu_0) \leq \lambda(X_0)$ for any feasible $\mu_0$.
With the constraint $\mu_0 \leq \lambda$, we get the inequality $\mu_0 \leq \lambda \Rightarrow \mu_0(X) = \mu_0(\text{spt } \mu_0) \leq \lambda(\text{spt } \mu_0) \leq \lambda(X_0)$, for any feasible $\mu_0$, then $p^* \leq \lambda(X_0)$.
However, by definition, $\lambda(X_0)$ is feasible, so $p^* \leq \lambda(X_0)$. Therefore, $p^* = \lambda(X_0)$.

□

Adding a slack variable $\hat{\mu} \in M(X)$, s.t.,

$$\lambda \geq \mu_0 \geq 0, \quad \mu_0 + \hat{\mu}_0 = \lambda, \mu_0 \geq 0, \quad \hat{\mu}_0 \geq 0 \tag{III.7}$$

Then:

$$\begin{aligned} p^* = \sup \quad & \mu_0(X) \\ \text{s.t.} \quad & \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 + \mathcal{L}'\mu \\ & \mu_0 + \hat{\mu}_0 = \lambda \\ & \mu \geq 0, \mu_0 \geq 0, \mu_T \geq 0, \hat{\mu}_0 \geq 0 \\ & \text{spt } \mu \subset [0,T] \times X \times U \\ & \text{spt } \mu_0 \subset X, \text{spt } \mu_T \subset X_T \\ & \text{spt } \hat{\mu}_0 \subset X \end{aligned} \tag{III.8}$$

It can be shown that the primal problem III.8 it is associated with the dual shown below.

$$\begin{aligned} d^* = \quad \inf \quad & \int_X w(x) d\lambda(x) \\ \text{s.t.} \quad & \mathcal{L}v(t,x,u) \leq 0, \quad && \forall (t,x,u) \in [0,T] \times X \times U \\ & w(x) \geq v(0,x) + 1, && \forall x \in X \\ & v(T,x) \geq 0, && \forall (t,x) \in X_T \\ & w(x) \geq 0, && \forall x \in X \end{aligned} \tag{III.9}$$

**Theorem III.3** (Strong duality in $\infty$-dim LP problem [2])**.** *There is no duality gap between the primal infinite-dimensional problem and the dual infinite-dimensional problem, i.e.,*

$$p^* = d^* \tag{III.10}$$

*Proof: [2]*

**Lemma III.4.** *Let $(v, w)$ be a feasible pair V.1, then $v(0,.) \geq 0$ on $X_0$, and $w \geq 1$ on $X_0$.*

*Proof.* Given any $x_0 \in X_0$, there exist $u(t)$ such that $x(t) \in X$, $u(t) \in U$, $\forall t \in [0,T]$ and $x(T) \in X_T$.
Note that

$$v(T,.) \geq 0 \text{ on } X_T \tag{III.11}$$
$$\mathcal{L}v \leq 0 \text{ on } [0,T] \times X \times U \tag{III.12}$$

and since $\frac{d}{dt}v(t,x(t)) = \frac{dv}{dt} + \frac{dv}{dx}\frac{dx}{dt} = \dot{v} + \nabla v \cdot f = \mathcal{L}v$, and obviously $\int_0^T \frac{d}{dt}v(t,x(t))dt = v(T,x(T)) - v(0,x_0)$, then we can obtain:

$$
\begin{aligned}
0 \le v(T,x(T)) = v(0,x_0) + \int_0^T \frac{d}{dt}v(t,x(t))dt \\
= v(0,x_0) + \int_0^T \mathcal{L}v(t,x(t),u(t))dt \le v(0,x_0) \le w(x_0) - 1
\end{aligned}
\tag{III.13}
$$

Therefore, $w(x_0) \ge 1$ and $v(0,x_0) \ge 0$, on $X_0$. □

By Theorem III.2, the primal is related to the integral:

$$
p^* = \int_X \mathbb{I}_{X_0}(x)d\lambda(x) \tag{III.14}
$$

Then, by Theorem III.3, there is no duality gap, i.e., $p^* = d^*$, and $\exists(v_k, w_k) \in C^1([0,T] \times X) \times C(X)$ feasible such that

$$
p^* = d^* = \lim_{k\to\infty} \int_X w_k(x)d\lambda(x) \tag{III.15}
$$

and it can be shown (by the previous Lemma) that

$$
\lim_{k\to\infty} \int_X \left(w_k(x) - \mathbb{I}_{X_0}(x)\right) d\lambda(x) = 0 \tag{III.16}
$$

concluding that $w_k \underset{L_1}{\to} \mathbb{I}_{X_0}$.

**Theorem III.5** (Convergence of $w_k$ to $\mathbb{I}_{X_0}$ [2])**.** *Given $w_k \in \mathbb{R}_{2k}[x]$ as an optimal solution to the dua LMI, then $w_k \searrow \mathbb{I}_{X_0}$ (converges from above) in $L^1$.*
*Proof: [2]*

It is shown in [2] that the $\infty$-dim LP problem can be approximated by a series of LMI problems, with vanishing error as the relaxation order increases.
Given the primal and dual, $p^*$ and $d^*$, which approximate the $\infty$-dim problem, one can see that the following sets are outer approximations of the ROA:

$$
X_{0k} := \{x \in X : v_k(0,x) \ge 0\} \quad \text{and} \quad \bar{X}_{0k} := \{x \in X_{\bar{v}_k}(0,x) \ge 0\} \tag{III.17}
$$

where $\bar{w}_k := \min_{i\ge k} w_i$ and $\bar{v}_k := \min_{i\ge k} v_i$, for polynomials $(w_k, v_k)$ with degree up to $2k$.

Furthermore, $X_{0k} \supset \bar{X}_{0k} \supset X_0$.

## IV. Convergence Analysis

**Corollary IV.1** (The sequence of LMI converge monotonically to the $\infty$-dim LP)**.**

$$
d^* \le d^*_{k+1} \le d^*_k \quad \textit{and} \quad p^* \le p^*_{k+1} \le p^*_k \tag{IV.1}
$$

*Proof.* From Theorem III.5 and relaxation of constraint with $k \to \infty$, we get convergence of dual optima $d^*_k$.
For the primal, weak duality: $d^*_k \ge p^*_k$, and from strong duality III.3: $d^* \to d^* = p^*$, then, $p^*_k \ge p^*$.
Also, the higher the relaxation order, the higher the constraints, so $p^*_{k+1} \le p^*_k$. □

**Theorem IV.2** (Convergence to ROA)**.** *Given $(v_k, w_k) \in \mathbb{R}_{2k}[t,x] \times \mathbb{R}_{2k}[x]$ be a solution to the dual LMI problem. Then, the sets $X_{0k}$ and $\bar{X}_{0k}$ converge to the ROA $X_0$ from the outside, such that*

$$
X_{0k} \supset \bar{X}_{0k} \supset X_0 \tag{IV.2}
$$

$$
\lim_{k\to\infty} \lambda(X_{0k} \setminus X_0) = 0 \textit{ and } \lim_{k\to\infty} \lambda(\bar{X}_{0k} \setminus X_0) = 0 \tag{IV.3}
$$

*Proof.* From Lemma III.4 and $\bar{X}_{0k} = \cup_{i=1}^k X_{0i}$ if follows that $X_{0k} \supset \bar{X}_{0k} \supset X_0$. Then, monotonicity of the sequence $\bar{X}_{0k}$ is implied.
Also from Lemma III.4, we have $w_k \ge \mathbb{I}_{X_0}$, and then since $w_k \ge v_k(0,.) + 1 \Rightarrow w_k \ge \mathbb{I}_{X_{0k}} \ge \mathbb{I}_{\bar{X}_{0k}} \ge \mathbb{I}_{X_0}$.
From Theorem III.5, $w_k \to \mathbb{I}_{X_0}$, then

$$
\lambda(X_0) = \int_X \mathbb{I}_{X_0} d\lambda \underbrace{=}_{\text{III.5}} \lim_{k\to\infty} \int_X w_k d\lambda \ge \lim_{k\to\infty} \int_X \mathbb{I}_{X_{0k}} d\lambda = \lim_{k\to\infty} \lambda(X_{0k}) \ge \lim_{k\to\infty} \lambda(\bar{X}_{0k}) \tag{IV.4}
$$

But $X_0 \subset \bar{X}_{0k} \subset X_{0k}$, then $\lambda(X_0) \le \lambda(\bar{X}_{0k}) \le \lambda(X_{0k})$. □

## V. Polynomial Optimization Problem

Utilizing the Weierstrass theorem, [3], one can approximate continuous functions with polynomials in a compact set. Furthermore, the domain sets are defined as semialgebraic Quadratic modules. The problem then yields a Polynomial Optimization problem (POP), which can be implemented and solved with a variety of computational tools, for example, with a the Mosek [1] optimization solver, as later will be shown in numerical examples.
The POP is formulated from the dual below.

$$\begin{aligned} &\inf \int_X w(x) d\lambda(x) \\ \text{s.t. } &\mathcal{L}v(t,x,u) \leq 0, && \forall (t,x,u) \in [0,T] \times X \times U \\ &w(x) \geq v(0,x) + 1, && \forall x \in X \\ &v(t,x) \geq 0, && \forall (t,x) \in [0,T] \times X_T \\ &w(x) \geq 0, && \forall x \in X \end{aligned} \tag{V.1}$$

Where, the constraints are written as nonnegativity certificates, which are related to sums of squares and quadratic modules.

$$\begin{aligned} w(x) - v(0,x) - 1 \geq 0, &\quad \forall x \in X \\ v(t,x) \geq 0, &\quad \forall (t,x) \in [0,T] \times X_T \\ w(x) \geq 0, &\quad \forall x \in X \end{aligned} \tag{V.2}$$

It is imperative to design suitable semialgebraic sets, then, given $X = [a,b]$, in order to find a semialgebraic domain such that

$$K = \{(t,x) \in \mathbb{R}^2 : g_1(t,x) \geq 0 \wedge g_2(t,x) \geq 0\} = [0,T] \times X \tag{V.3}$$

a simple concave quadratic polynomial is a possible representation, obtaining:

$$K = \{(t,x) \in \mathbb{R}^2 : -t(t-T) \geq 0 \wedge -(x-a)(x-b) \geq 0\} = [0,T] \times [a,b] \tag{V.4}$$

Similarly, for the remaining domains:

$$X = \{x \in \mathbb{R} : -(x-a)(x-b) \geq 0\} \tag{V.5}$$

$$K_T = \{(t,x) \in \mathbb{R}^2 : -t(t-T) \geq 0 \wedge -(x-\alpha)(x-\beta) \geq 0\} = [0,T] \times X_T \tag{V.6}$$

## VI. Numerical Simulations

All implementations are available at `https://github.com/pedroblossbraga/ROA_computation/tree/main`.

### A. Univariate Cubic Dynamics

Consider the following simple uncontrolled univariate cubic example with dynamics:

$$\dot{x} = x(x-0.5)(x+0.5) \tag{VI.1}$$

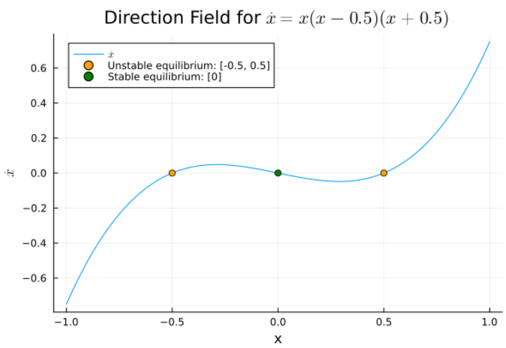


FIG. 1: ROA approximations for $d \in \{4, 6, 16, 32\}$.

Additionally, taking $X = [-1, 1]$, $T = 1$, $X_T = [-0.24, 0.24]$, the following numerical results are obtained, increasing the velocity of the system with a factor of $10$, i.e., $\tilde{f} = 10f$:

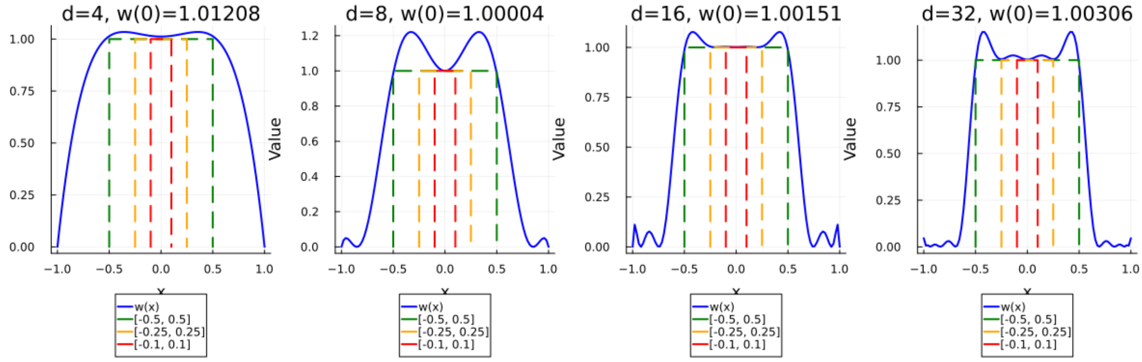


FIG. 2: ROA approximations for $d \in \{4, 6, 16, 32\}$.

Note that $w(x)$ evaluated at the stable equilibrium point $x = 0$ still satisfies the constraint $w(x) \geq 1$. Furthermore, the ROA appears to be located in the interval $[-0.5, 0.5]$, with a Gibbs effect in the discontinuity neighborhood, in $\{-0.5, 0.5\}$.
And, by reversing the ODE in time, with an initial condition of $x(0) = 0.24$, it is possible to verify if the value matches the observed ROA interval of $[-0.5, 0.5]$. As the system is symmetric, it is not necessary to verify for the left bound.

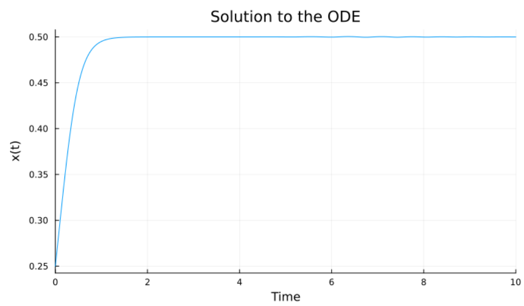


FIG. 3: Reversed ODE solution

The ODE solution converges to the value of $0.5$.

## VII. References


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