Friedrich-Alexander-Universität
Erlangen-Nürnberg

FAU

Existence of Optimal Controls for Renormalized Solutions of Nonlinear Evolution Equations

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1. Nonlinear evolution equations
2. Assumptions
3. Well-posedness of PDE
4. Existence of optimal controls

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Nonlinear Evolution Equations

$$\begin{aligned} \partial_t y - \operatorname{div}\big(a(t,x,y,Dy)\big) + \operatorname{div}(\Phi(y)) &= u && \text{in } (0,T)\times\Omega, \\ y &= 0 && \text{on } (0,T)\times\partial\Omega, \\ y(t=0) &= y_0 && \text{in } \Omega. \end{aligned} \tag{Pb 1}$$

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- $-\operatorname{div}\big(a(t,x,y,Dy)\big) : L^q(0,T;W_0^{1,q}(\Omega)) \to L^{q'}(0,T;W^{-1,q'}(\Omega))$ is a Leray-Lions operator, coercive, monotone, satisfying a standard growth condition, and $q>1$ is real.

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- $\Phi : \mathbb{R} \to \mathbb{R}^N$ is assumed to be continuous.
- $u \in L^1(Q)$ and $y_0 \in L^1(\Omega)$.

Truncation process

Given $K > 0$, $T_K(s) := \max(-K, \min(s, K))$.

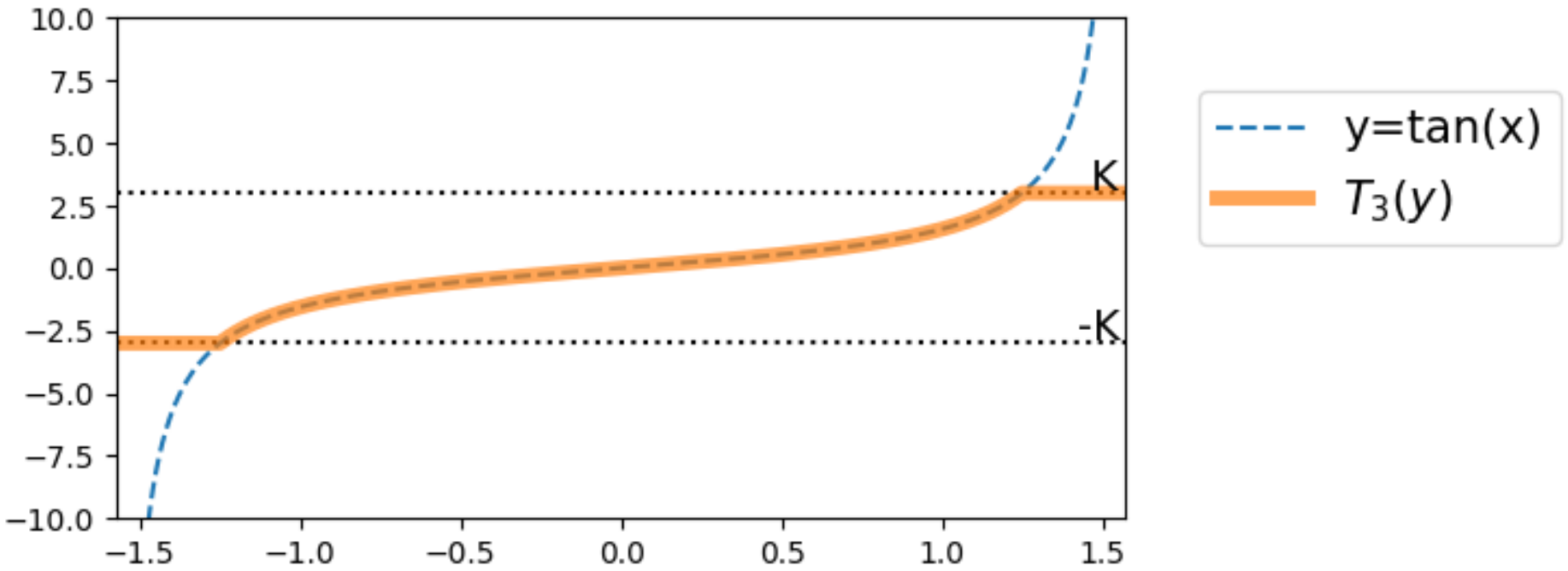


Figure: Example of truncation with $y = \tan(x)$, and $T_3(y)$

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$$\int_{\{(t,x)\in Q; n\leq |y(t,x)|\leq n+1\}} a(t,x,y,Dy)\cdot Dy \,\mathrm{d}x\mathrm{d}t \to 0 \text{ as } n \to \infty, \tag{1.2}$$

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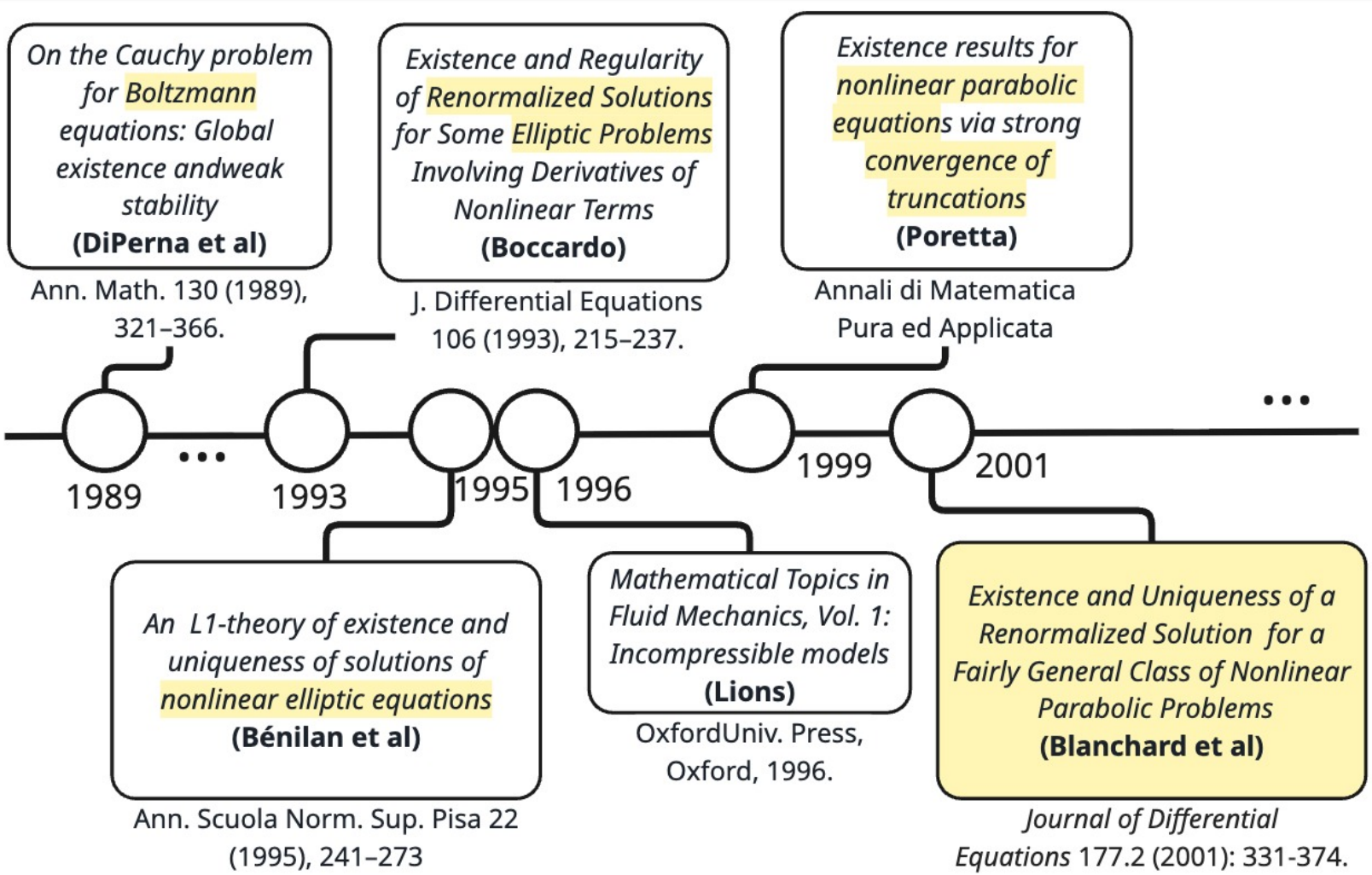
$$\int_{\{(t,x)\in Q; n\le|y(t,x)|\le n+1\}} a(t,x,y,Dy)\cdot Dy \,\mathrm{d}x\mathrm{d}t \to 0 \text{ as } n \to \infty, \tag{1.2}$$

For every $S \in W^{2,\infty}(\mathbb{R})$ with S' compactly supported,

$$\begin{aligned} \partial_t S(y) - \operatorname{div}\big(S'(y)a(y,Dy)\big) + S''(y)a(y,Dy)\cdot Dy \\ + \operatorname{div}\big(S'(y)\Phi(y)\big) - S''(y)\Phi(y)\cdot Dy = uS'(y). \end{aligned} \tag{1.3}$$

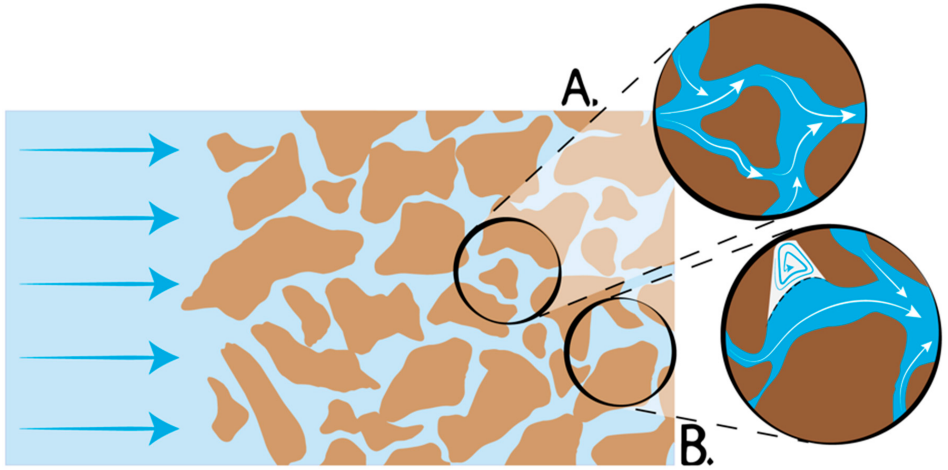
$$S(y)(t=0) = S(y_0) \text{ in } \Omega. \tag{1.4}$$

Some previous works



Applications

- Nonlinear diffusion problems (Kefi, Yang, Hamdani, and Liu 2025);
- Porous media (Abdellaoui and Redwane 2023a; Abdellaoui and Redwane 2023b);
- Image processing;
- Energy-transfer (e.g.Semiconductors) Bessemoulin-Chatard, Lissoni, and Mathis 2026;
- *et cetera*.



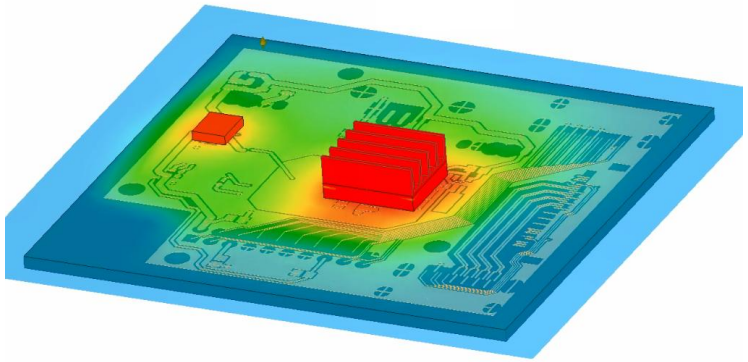


Figure: (*Icepak* | *Macnica Galaxy* 2026; Young and Kabala 2024)

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2. **Assumptions**
3. Well-posedness of PDE
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Assumptions

$$\partial_t y - \operatorname{div}\big(a(t,x,y,Dy)\big) + \operatorname{div}(\Phi(y)) = u \tag{2.1}$$

Assumptions (Blanchard, Murat, and Redwane 2001)

Consider $1 < q < \infty$,

- $Q = (0,T) \times \Omega$ where Ω is a bounded subdomain of $\mathbb{R}^N$ ($N \geq 1$),

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- $a(t,x,s,\xi) \cdot \xi \geq \mu |\xi|^q$, for a.e. $(t,x) \in Q, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \mu \in \mathbb{R}^+$,

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- $a(t,x,s,\xi) \cdot \xi \geq \mu |\xi|^q$,
- For any $K > 0$, $\exists b \in L^{q'}(Q), C_K > 0$ s.t.

$$|a(t,x,s,\xi)| \leq C_K\big(b(t,x) + |\xi|^{q-1}\big),$$

for a.e. $(t,x) \in Q, \forall s : |s| \leq K$, and $\forall \xi \in \mathbb{R}^N$,

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Existence and Uniqueness of renormalized solutions for (Pb 1)

Theorem 1 (Existence Blanchard, Murat, and Redwane 2001)

Let $u \in L^1(Q)$ *and* $y_0 \in L^1(\Omega)$. *There* ***exists a renormalized solution*** *to problem* (Pb 1).

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- *local L-continuity of Φ,*
- *for every $K > 0$, $\exists \tau_K \geq 0, E_K \in L^{q'}(Q)$, **s.t.,***

$$\left|a(t,x,s,\xi) - a(t,x,s',\xi)\right| \leq |s - s'|\left[E_K(t,x) + \tau_K|\xi|^{q-1}\right]$$

*for a.e. (t,x) in Q, $\forall s, s'$ **s.t.** $|s|, |s'| \leq K$, $\forall \xi \in \mathbb{R}^N$,*

*the renormalized solution to problem (Pb 1) is **unique**.*

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Note: Uniqueness under *local regularity assumptions* on a and Φ!

Blanchard, Murat, and Redwane (2001)
Stability Result

Theorem 3 (Blanchard, Murat, and Redwane 2001)

$$u_n \rightharpoonup u \quad \text{in } L^1(Q)$$
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Let y_n be renorm. sol. for (Pb 1). For any $K \geq 0$,

$$T_K(y_n) \rightharpoonup T_K(y) \quad \text{in } L^q(0,T;W_0^{1,q}(\Omega)) \tag{T3}$$

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Moreover, if a is strictly monotone, then the convergence in (T3) is strong.

Optimal control problem

$$\underset{y,u}{\text{minimize}} \quad J(y,u) := \frac{\gamma}{2}\int_0^T \int_\Omega |y(x,t) - y_d(x,t)|^2 \,\mathrm{d}x\mathrm{d}t + \frac{\gamma_T}{2}\int_\Omega |y(x,T) - y_T(x)|^2 \,\mathrm{d}x + \frac{\alpha}{2}\|u\|^2_{L^2(Q)}$$

$$\text{subject to} \quad \partial_t y - \operatorname{div}(a(t,x,y,Dy)) + \operatorname{div}(\Phi(y)) = u \quad \text{in } (0,T)\times\Omega$$

$$y = 0 \quad \text{on } (0,T)\times\partial\Omega,$$

$$y(t=0) = y_0 \quad \text{in } \Omega,$$

$$u \in \mathcal{U}_{\text{ad}} := \left\{u \in L^2(Q) : 0 \le u \le u_{\text{max}} \text{ a.e. in } Q\right\}. \tag{OCP}$$

With $\gamma, \gamma_T \ge 0$, and $\alpha > 0$.

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2. Γ-convergence: $J_K \xrightarrow{\Gamma} J$

Existence of optimal controls for (OCP)

$$J(y(u),u) = \frac{\gamma}{2}\|y(u) - y_d\|^2_{L^2(Q)} + \frac{\gamma_T}{2}\|y(u)(T) - y_T\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(Q)}. \qquad \text{(Cost)}$$

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$$J_K(y(u),u) := \frac{\gamma}{2}\|T_K\big(y(u)\big) - T_K(y_d)\|^2_{L^2(Q)} + \frac{\gamma_T}{2}\|T_K\big(y(u)(T)\big) - T_K(y_T)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(Q)}.$$

(Truncated Cost)

Existence of optimal controls for (OCP)

1) Direct method, minimizer $J_K(u_*)$

Show: exists a minimizer $J_K(u_*^K)$ $(\exists u_*^K \in \mathcal{U}_{\text{ad}} : J_K(u_*^K) \leq J_K(u), \forall u \in \mathcal{U}_{\text{ad}})$.

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- Stability (T3) $\Rightarrow T_K(y(u_n^K)) \rightharpoonup T_K(y(u_*^K))$ in $L^2(Q)$

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- Tracking w.l.s.c $\liminf_{n\to\infty} \|T_K\big(y(u_n^K)\big) - T_K(y_d)\|^2_{L^2(Q)} \geq \|T_K\big(y(u_*^K)\big) - T_K(y_d)\|^2_{L^2(Q)}$

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- $u \mapsto \frac{\alpha}{2}\|u\|^2_{L^2(Q)}$ is convex and continuous, hence, w.l.s.c.

Existence of optimal controls for (OCP)

1) Direct method, minimizer $J_K(u_*)$

$$J_K(u_*^K) = \underbrace{\frac{\gamma}{2}\|T_K\big(y(u_*^K)\big) - T_K(y_d)\|^2_{L^2(Q)} + \frac{\gamma_T}{2}\|T_K\big(y(u_*^K)(T)\big) - T_K(y_T)\|^2_{L^2(\Omega)}}_{\text{w.l.s.c or continuous}}$$

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- Fix $M > 0 : M \leq K_j$,

$$\|T_{K_j}\big(y(u_*^{K_j})\big) - T_{K_j}(y_d)\|^2_{L^2(Q)} \geq \|T_M\big(y(u_*^{K_j})\big) - T_M(y_d)\|^2_{L^2(Q)}$$

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- By stability (T3), $T_M\big(y(u_*^{K_j})\big) \to T_M\big(y(u_*)\big)$ in $L^2(Q)$ when $j \to \infty$, so

$$\liminf_{j\to\infty} \|T_{K_j}\big(y(u_*^{K_j})\big) - T_{K_j}(y_d)\|^2_{L^2(Q)} \geq \|T_M\big(y(u_*)\big) - T_M(y_d)\|^2_{L^2(Q)}.$$

Existence of optimal controls for (OCP)

2) $J(u^*) \leq \liminf_{K\to\infty} J_K(u^*)$

- let $M \to \infty$ and use $\lim_{K\to\infty} \|y - T_K(y)\|^2_{L^2(Q)} = 0$. Then,

$$\liminf_{j\to\infty} \|T_{K_j}\left(y(u_*^{K_j})\right) - T_{K_j}(y_d)\|^2_{L^2(Q)} \geq \|y(u_*) - y_d\|^2_{L^2(Q)}.$$

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- w.l.s.c. of regularization + l.s.c. of cost,

$$J(u_*) \leq \liminf_{j\to\infty} J_K(u_*^{K_j}).$$

Existence of optimal controls for (OCP)

2) $J(u^*) \geq \limsup_{K\to\infty} J_K(u^*)$

- Taking a *recovery sequence* $u^K = u_*$, since $T_K\big(y(u^K)\big) = T_K\big(y(u_*)\big) \to y(u_*)$ and $T_K(y_d) \to y_d$ in $L^2(Q)$ as $K \to \infty$, then

$$\limsup_{K\to\infty} J_K(u_*^K) \leq \lim_{K\to\infty} J_K(u_*) = J(u_*).$$

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$$\limsup_{K\to\infty} J_K(u_*^K) \leq \lim_{K\to\infty} J_K(u_*) = J(u_*).$$

Therefore,

$$\limsup_{K\to\infty} J_K(u_*^K) \leq J(u_*) \leq \liminf_{K\to\infty} J_K(u_*^K). \quad \square$$

Optimality of u_*

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- $J(u_*) \leq \liminf_{K_j \to \infty} J_{K_j}(u_*^{K_j}) \leq \limsup_{K_j \to \infty} J_{K_j}(u_*^{K_j}) \leq J(v)$.

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- $J(u_*) \leq \liminf_{K_j \to \infty} J_{K_j}(u_*^{K_j}) \leq \limsup_{K_j \to \infty} J_{K_j}(u_*^{K_j}) \leq J(v)$.
- $\Rightarrow J(u_*) \leq J(v), \quad \forall v \in \mathcal{U}_{\text{ad}}$.

Existence of optimal controls for (OCP)

$J_K \xrightarrow{\Gamma} J$, and since (u_*^K) is a minimizing sequence of J_K, every cluster point of the sequence is a minimizer of J, provided equicoercivity of (J_K).

Existence of optimal controls for (OCP)

$J_K \xrightarrow{\Gamma} J$, and since (u_*^K) is a minimizing sequence of J_K, every cluster point of the sequence is a minimizer of J, provided equicoercivity of (J_K).

Thus, there exists a $u^* \in \mathcal{U}_{\text{ad}}$ that minimizes J.

Final remarks

Remark 1

The existence result shown can be generalized for datum $u \in L^p(Q)$ and $y_0 \in L^p(\Omega)$, with $1 \leq p < q < \infty$.

Further work:

- Optimality conditions
- Numerics

Thank you! Any questions or comments?

Renormalized Solutions in der Optimalsteuerung von Evolutionsgleichungen DFG project n.551486487, supervised by Prof. Dr. Hannes Meinlschmidt.





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