Friedrich-Alexander-Universität
Erlangen-Nürnberg

Optimizing bounds for energy-constrained cooling in 2D

Pedro Blöss Braga[1] **Giovanni Fantuzzi**[1]

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1. Introduction to optimal cooling
2. Lower bounds on optimal cooling
3. Efficiency bounds
4. SDP discretization
5. Numerical examples
6. Improved bounds
7. Conclusions

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Recent work (Braga and Fantuzzi 2026)

OPTIMIZING BOUNDS FOR ENERGY-CONSTRAINED OPTIMAL COOLING PROBLEMS IN TWO DIMENSIONS

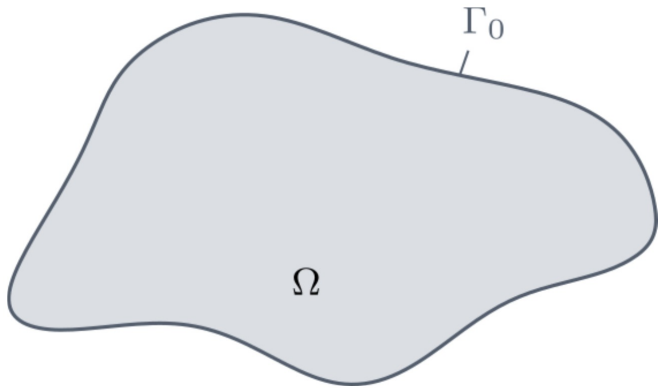
PEDRO BLÖSS BRAGA AND GIOVANNI FANTUZZI

Department of Mathematics
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ABSTRACT. We study optimal control problems for incompressible fluids in two-dimensional domains with a cold boundary and internal heat sources and sinks. Given a kinetic energy budget, measured by the square of a nondimensional Péclet number Pe, the goal is to maximize a cooling efficiency inversely proportional to the mean square gradient of the fluid's temperature. Using Lagrange duality, we formulate a well-posed dual problem whose solution yields an upper bound on the maximum cooling efficiency $\mathcal{E}(\mathrm{Pe})$. We then numerically approximate the dual problem using a convergent hierarchy of semidefinite programs obtained via discretization. We illustrate this computational approach on optimal cooling problems in a square and in an annulus, explaining also how problem symmetries can be exploited to reduce computational complexity. Finally, we construct admissible points for the dual problem to obtain new analytical upper bounds on the optimal cooling efficiency $\mathcal{E}(\mathrm{Pe})$. Specifically, we prove that $\mathcal{E}(\mathrm{Pe}) \lesssim \mathrm{Pe}^2$ for arbitrary domains and heat distributions, and that $\mathcal{E}(\mathrm{Pe}) \lesssim \mathrm{Pe}^2/\ln^2 \mathrm{Pe}$ for cooling flows in disks and annuli with heat source/sink distributions with a positive azimuthal average. These results generalize and improve known efficiency bounds for energy-constrained cooling flows in a disk.

Figure: https://arxiv.org/abs/2608.14334

Optimizing heat transfer in 2D



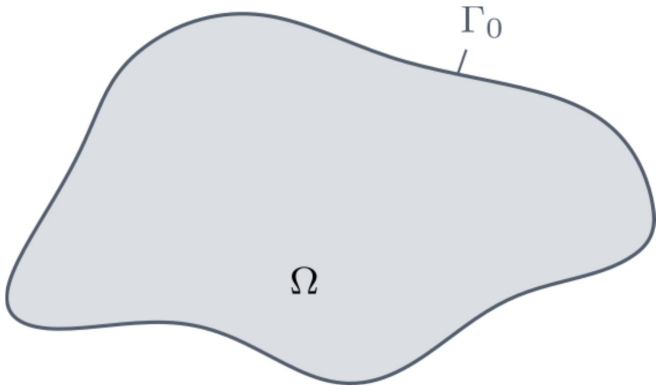
Optimizing heat transfer in 2D

$$\mathcal{J}(\mathsf{Pe}) = \min_{\substack{u\in[L^2(\Omega)]^2 \\ T\in H_0^1(\Omega)}} \int |\nabla T|^2 \,\mathrm{d}x$$

subject to

$$\begin{aligned}
&\operatorname{div} u = 0 && \text{in } \Omega \\
&u \cdot \hat{n} = 0 && \text{on } \partial\Omega \\
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&\int |u|^2 \,\mathrm{d}x \leq \mathsf{Pe}^2
\end{aligned}$$

(OCP)



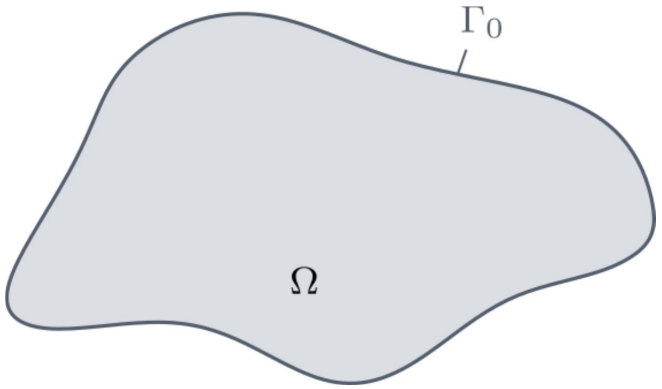
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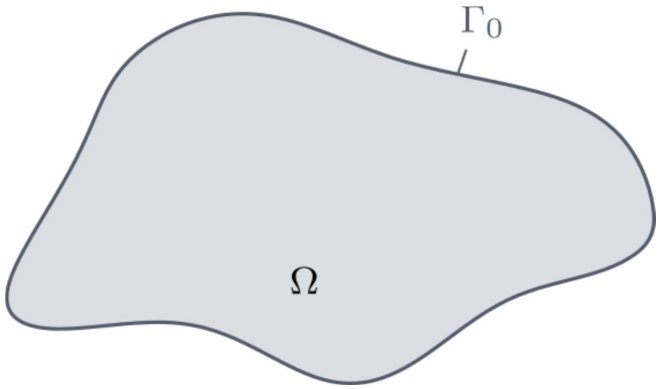
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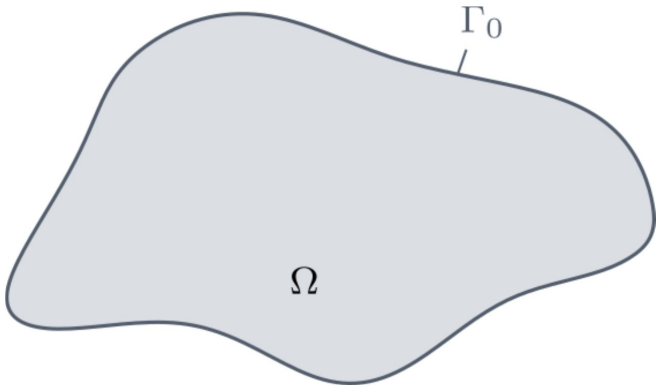
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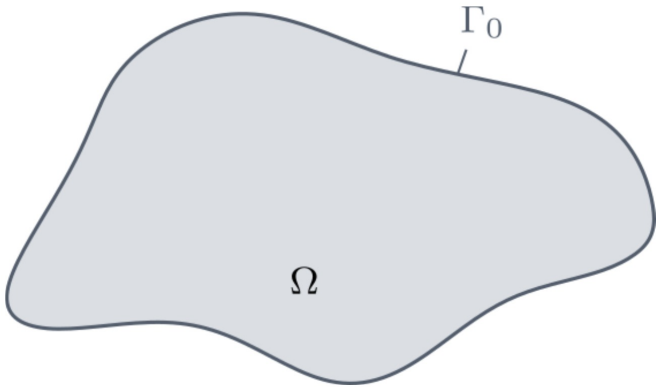
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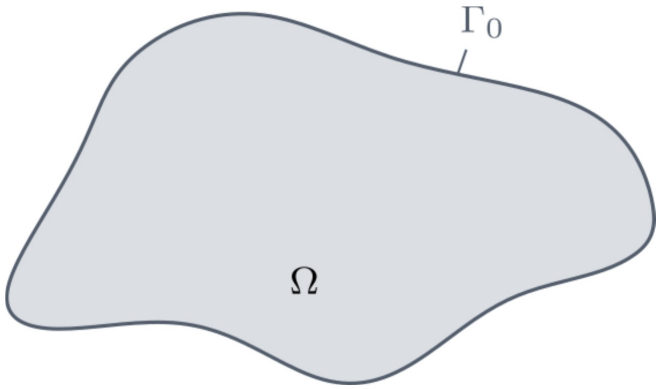
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Express u in terms of a streamfunction $\psi \in H_0^1(\Omega)$ vanishing on $\partial\Omega$,

$$u = \mathrm{Pe}\,\nabla^\perp\psi = \mathrm{Pe}(\partial_y\psi - \partial_x\psi)$$

Optimizing heat transfer in 2D

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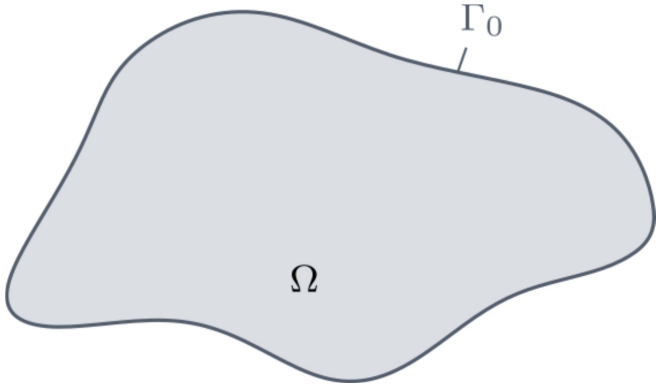
subject to

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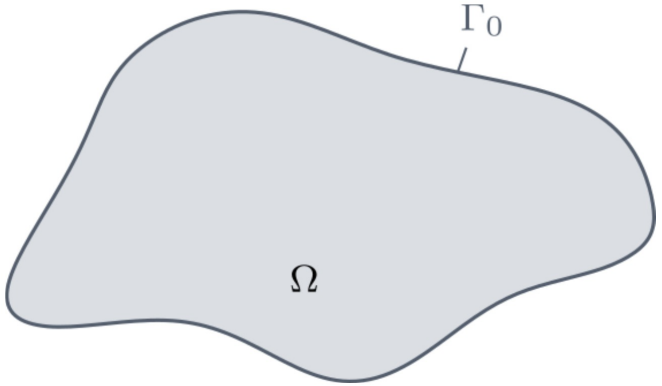


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Relation to *efficiency* $\mathcal{E}(\mathrm{Pe})$

Given an energy budget Pe^2, maximize $\mathcal{E}(\mathrm{Pe}) \propto \dfrac{1}{|\nabla T|^2}$.

Previous works

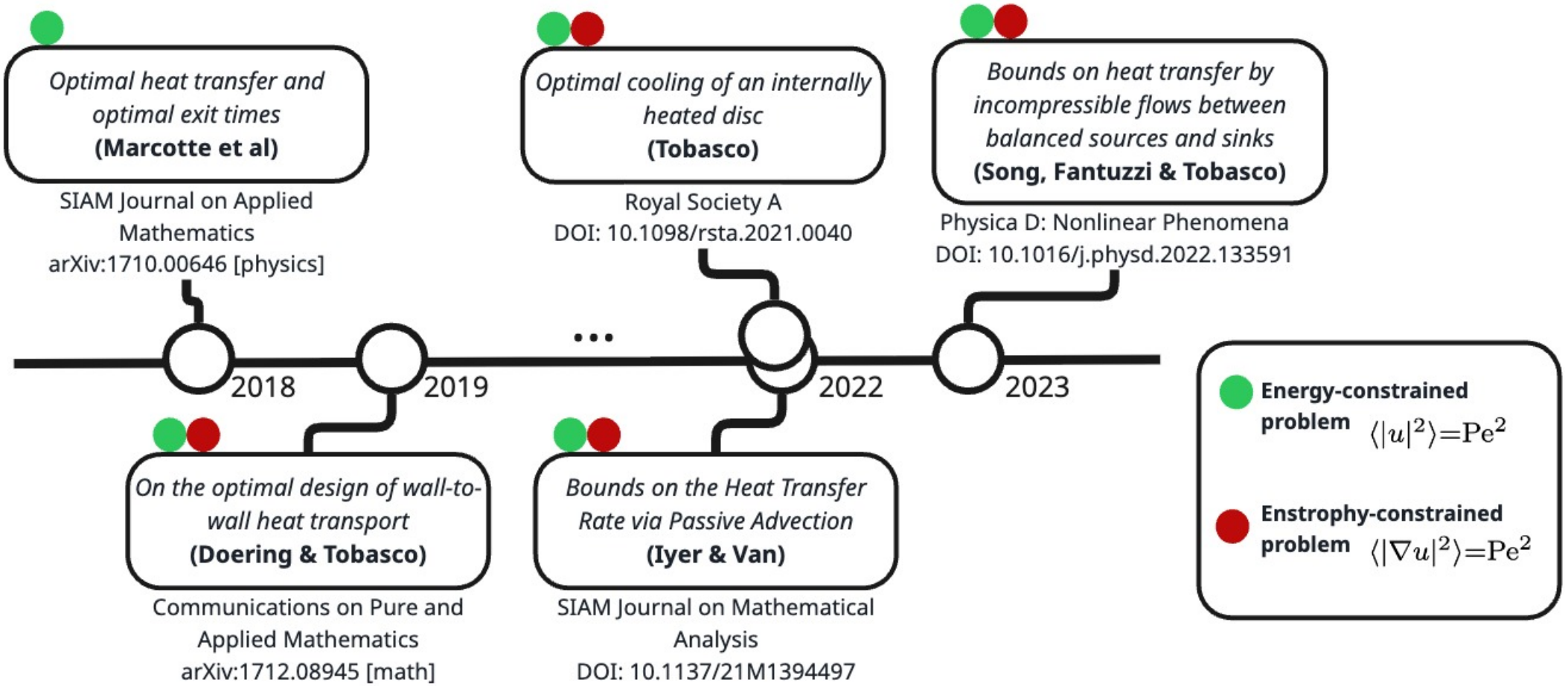


Figure: Timeline of previous works.

Previous works

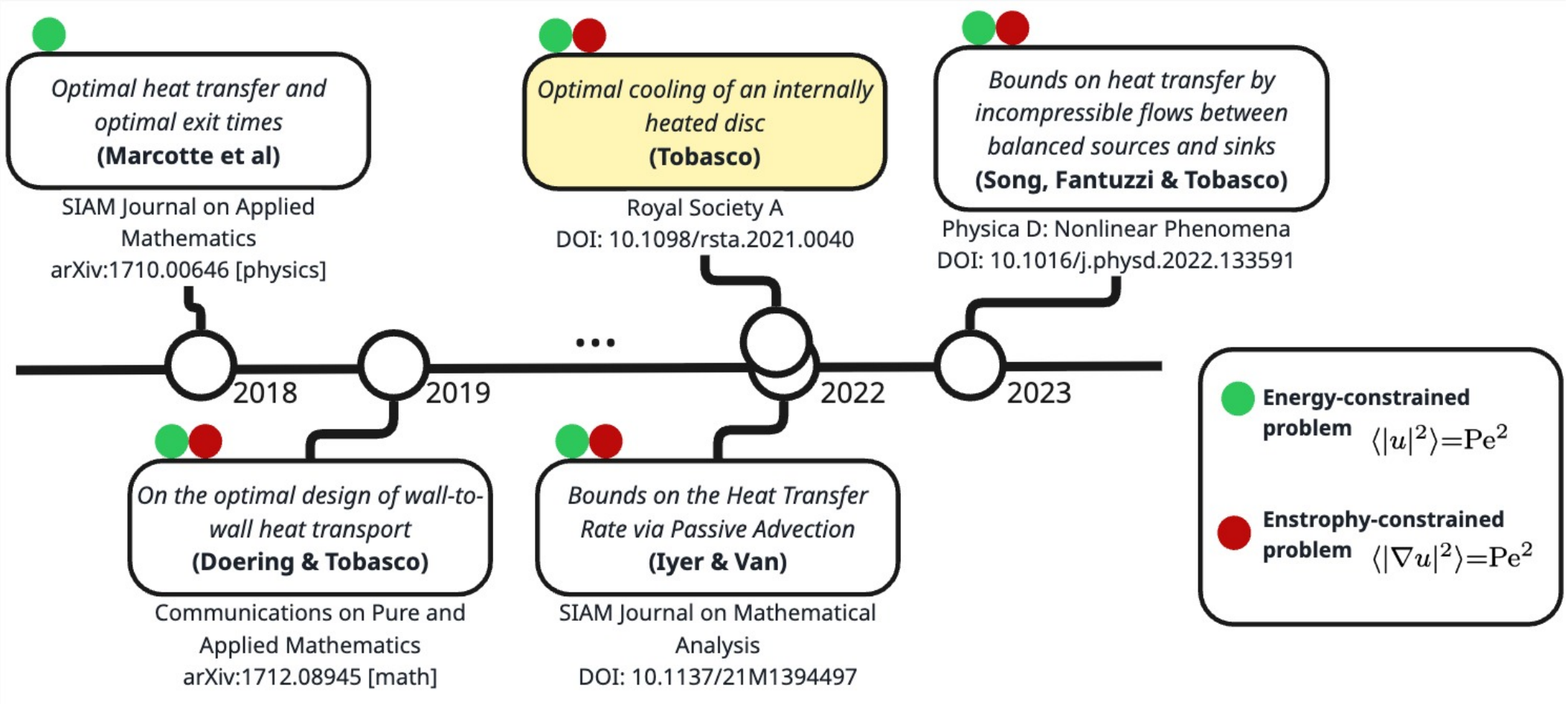


Figure: Timeline of previous works.

Tobasco (2022) : Optimal cooling in a disk

- **Enstrophy-constrained** problem: $\exists C(f), C'(f), c(f) > 0$ s.t.,

$$C\frac{1}{\mathsf{Pe}^{2/3}} \leq \min_{\substack{\langle|\nabla u|^2\rangle \leq \mathsf{Pe}^2 \\ \dots}} \langle|\nabla T|^2\rangle \leq C'\frac{\log^{4/3}\mathsf{Pe}}{\mathsf{Pe}^{2/3}}, \quad \forall \mathsf{Pe} \geq c(f).$$

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- **Energy-constrained**: $\exists C, C' > 0$, s.t.,

$$\frac{C}{\mathsf{Pe}^2} \leq \min_{\substack{\langle|u|^2\rangle \leq \mathsf{Pe}^2 \\ \dots}} \langle|\nabla T|^2\rangle \leq \frac{C'}{\mathsf{Pe}}\langle|f|^2 + |\nabla f|^2\rangle, \quad \forall \mathsf{Pe} \geq 1.$$

Optimal cooling efficiency (Braga and Fantuzzi 2026)

- Find a systematic way to **optimize efficiency bounds**

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  - $\mathcal{E}(\mathrm{Pe}) \lesssim \mathrm{Pe}^2$ (arbitrary domains)
  - $\mathcal{E}(\mathrm{Pe}) \lesssim \dfrac{\mathrm{Pe}^2}{\ln^2(\mathrm{Pe})}$ (disks and annuli)



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Lagrange dual ∞-dim problem

$$\mathcal{J}(\mathrm{Pe}) = \min_{\substack{\psi \in H_0^1(\Omega) \\ T \in H_0^1(\Omega)}} \int |\nabla T|^2 \, \mathrm{d}\mathbf{x} \quad \text{s.t. } -\Delta T + \nabla^\perp \psi \cdot \nabla T = f \text{ and } \langle |\nabla \psi|^2 \rangle \leq \mathrm{Pe}^2$$

- Lagrange multiplier $a \in \mathbb{R}$ for the kinetic energy constraint
- Lagrange multiplier $\xi \in H_0^1(\Omega)$ for the advection-diffusion equation

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$$\geq \max_{\substack{\xi \in H_0^1(\Omega) \\ a \in \mathbb{R}}} \min_{\substack{u \in \mathcal{U} \\ T \in H_0^1(\Omega)}} \int |\nabla T|^2 + a|\nabla \psi|^2 + \mathrm{Pe}\, \nabla \xi \cdot \nabla^\perp \psi T + \xi f - \frac{1}{4}|\nabla \xi|^2 \,\mathrm{d}\mathbf{x} - a$$

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$$\ge \max_{\substack{\xi\in H_0^1(\Omega)\\ a\in\mathbb{R}}} \underbrace{\min_{\substack{u\in\mathcal{U}\\ T\in H_0^1(\Omega)}} \int |\nabla T|^2 + a|\nabla\psi|^2 + \mathrm{Pe}\,\nabla\xi\cdot\nabla^\perp\psi T \,\mathrm{d}\mathbf{x}}_{\equiv\mathcal{Q}(a,\xi)[\psi,T]\succeq 0} + \int \xi f - \frac{1}{4}|\nabla\xi|^2 \,\mathrm{d}\mathbf{x} - a$$

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Lower bound on optimal cooling

Theorem 1

For every $\mathrm{Pe} \geq 0$,

$$\mathcal{J}(\mathrm{Pe}) \geq \mathcal{L}(\mathrm{Pe}),$$

where

$$\mathcal{L}(\mathrm{Pe}) = \max_{\substack{\xi \in H_0^1(\Omega) \\ a \in \mathbb{R}}} \int \xi f - \frac{1}{4}|\nabla \xi|^2 \,\mathrm{d}\mathbf{x} - a$$

$$\text{s.t.} \quad \int a|\nabla \psi|^2 + |\nabla T|^2 + \mathrm{Pe}\nabla \xi \cdot \nabla^{\perp} \psi T \,\mathrm{d}\mathbf{x} \geq 0, \quad \forall \psi, T \in H_0^1(\Omega). \tag{LB}$$

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- The term $\int \nabla \xi \cdot \nabla^\perp \psi T \,\mathrm{d}\mathbf{x}$ is problematic for L^2 functions as input.
- Remedy: Use Hardy and BMO spaces.

Lower bound on optimal cooling

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$$\textit{s.t.} \quad \int a|\nabla \psi|^2 + |\nabla T|^2 + \mathrm{Pe}\nabla \xi \cdot \nabla^{\perp} \psi T \,\mathrm{d}\mathbf{x} \geq 0, \quad \forall \psi, T \in H_0^1(\Omega).$$

Moreover, the dual problem (LB) has a **unique solution** $(a^*, \xi^*) \in H_0^1(\Omega) \times \mathbb{R}$.



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Efficiency bounds for arbitrary domains

Theorem 2

Let $c = c(\Omega)$. *For every* $\mathrm{Pe} \geq 0$,

$$\mathcal{J}(\mathrm{Pe}) \geq \mathcal{L}(\mathrm{Pe}) \geq \frac{\|f\|_{H^{-1}}^2}{1 + c^2\mathrm{Pe}^2}.$$

Hence, $\mathcal{E} \leq 1 + c^2\mathrm{Pe}^2$.



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Discretization and SDP formulation

$$\mathcal{L}(\mathrm{Pe}) \quad = \quad \max_{\substack{\xi \in H_0^1(\Omega) \\ a \in \mathbb{R}}} \quad \int \xi f - \frac{1}{4}|\nabla \xi|^2 \, \mathrm{d}\mathbf{x} - a$$

$$\text{s.t.} \quad \int a|\nabla \psi|^2 + |\nabla T|^2 + \mathrm{Pe}\nabla \xi \cdot \nabla^{\perp} \psi \, T \, \mathrm{d}\mathbf{x} \geq 0, \quad \forall \psi, T \in H_0^1(\Omega)$$

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Theorem 3 (Convergence of discretizations)

Fix $\mathrm{Pe} \geq 0$. *If* $\mathcal{H}_n$ *is 'asymptotically dense', then as* $n \to \infty$,

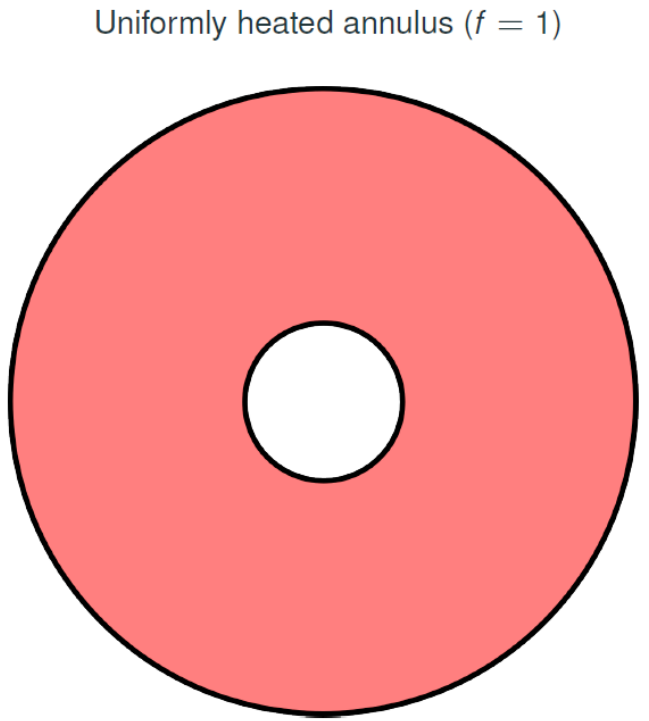
1. $\mathcal{L}_n(\mathrm{Pe}) \to \mathcal{L}(\mathrm{Pe})$
2. $(a_n^*, \xi_n^*) \to (a^*, \xi^*)$ *strongly in* $\mathbb{R} \times H_0^1(\Omega)$



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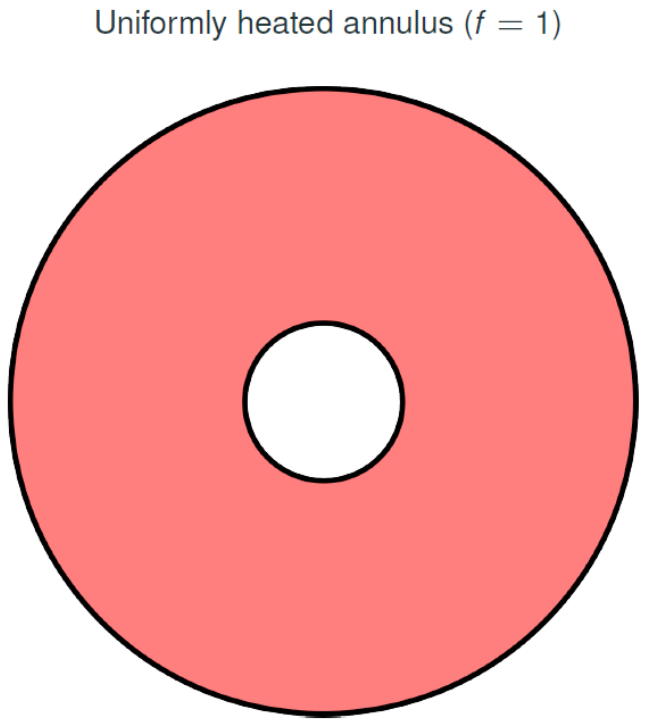
Uniformly heated annulus setting

- $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \ \rho < |\mathbf{x}| < 1\}$ with $\rho = \frac{1}{4}$



Uniformly heated annulus setting

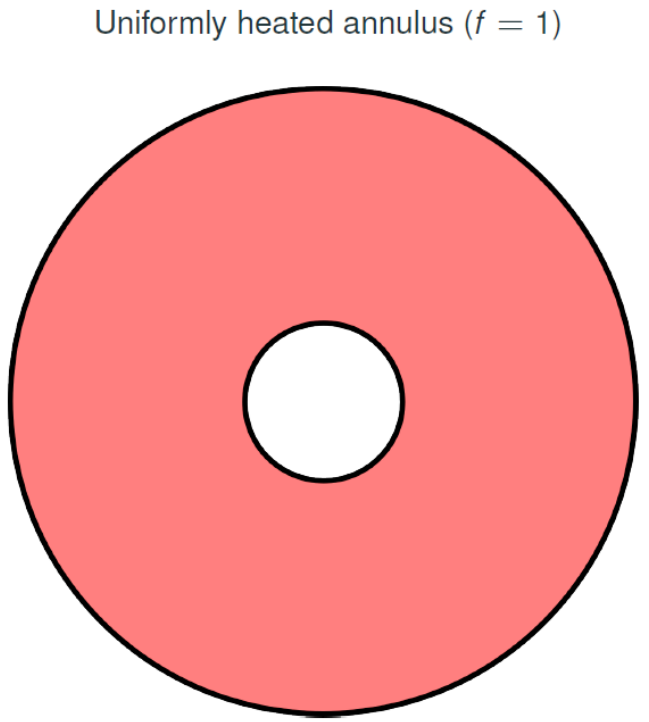
- $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \ \rho < |\mathbf{x}| < 1\}$ with $\rho = \frac{1}{4}$
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$$\mathcal{L}(\mathrm{Pe}) = \max_{\substack{a \in \mathbb{R} \\ \xi \in H_0^1(\rho,1) \\ \mathcal{Q}_k(a,\xi) \succeq 0 \, \forall k \in \mathbb{Z}}} 2\pi \int_\rho^1 \left(f\xi - \frac{1}{4} |\xi'|^2 \right) r \mathrm{d}r - a,$$

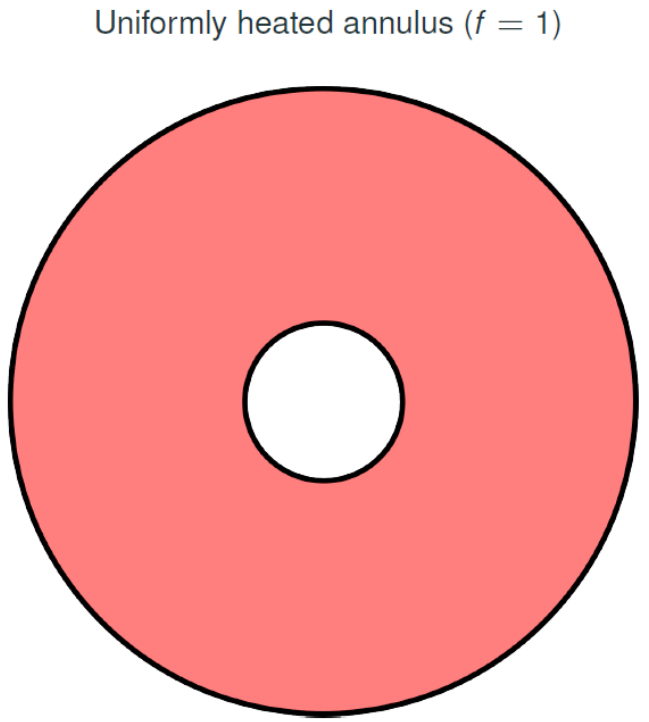


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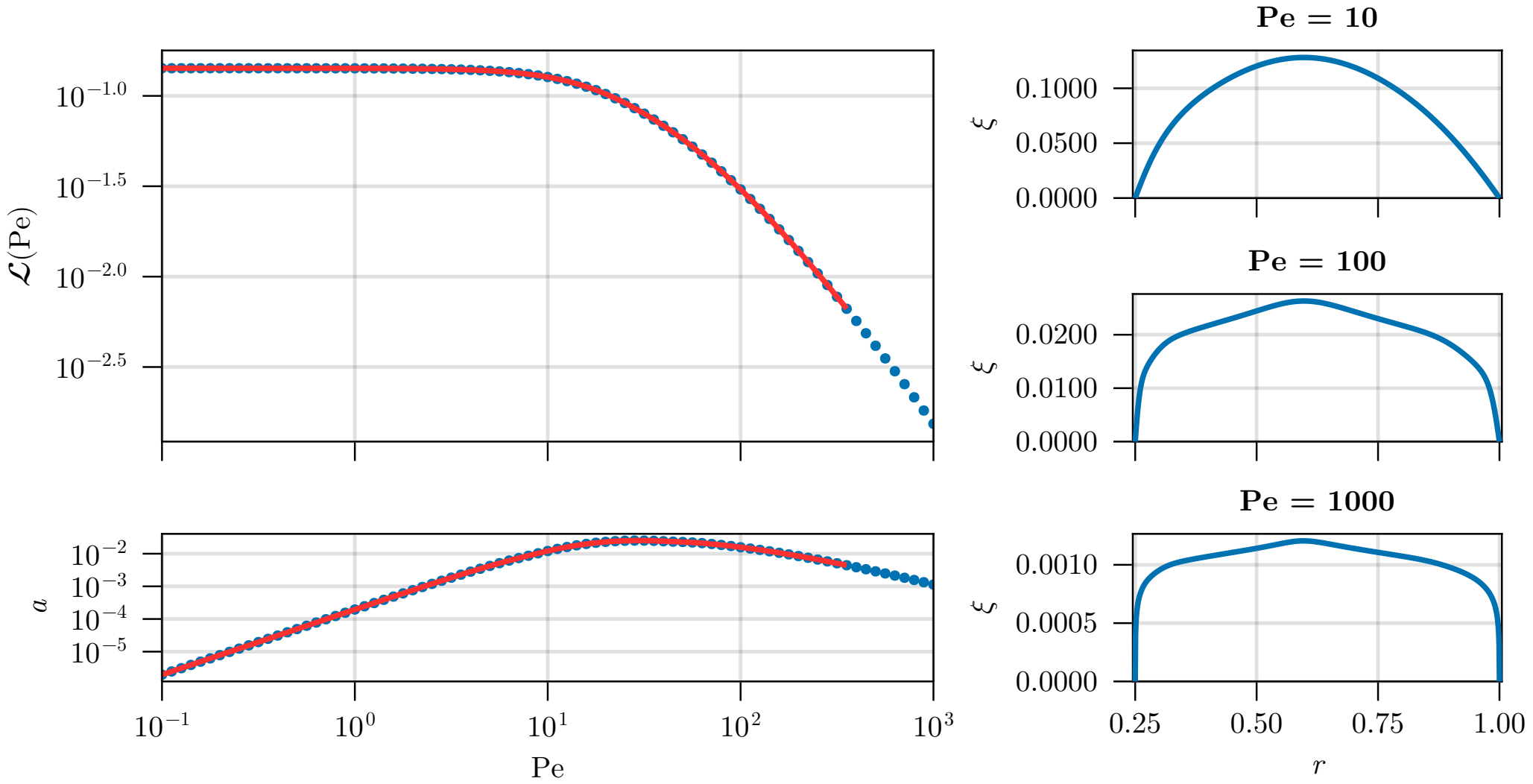
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- k: wavenumber of each Fourier mode



Optimal solutions of SDP discretizations for a uniformly heated annulus



Annulus example - active wavenumbers

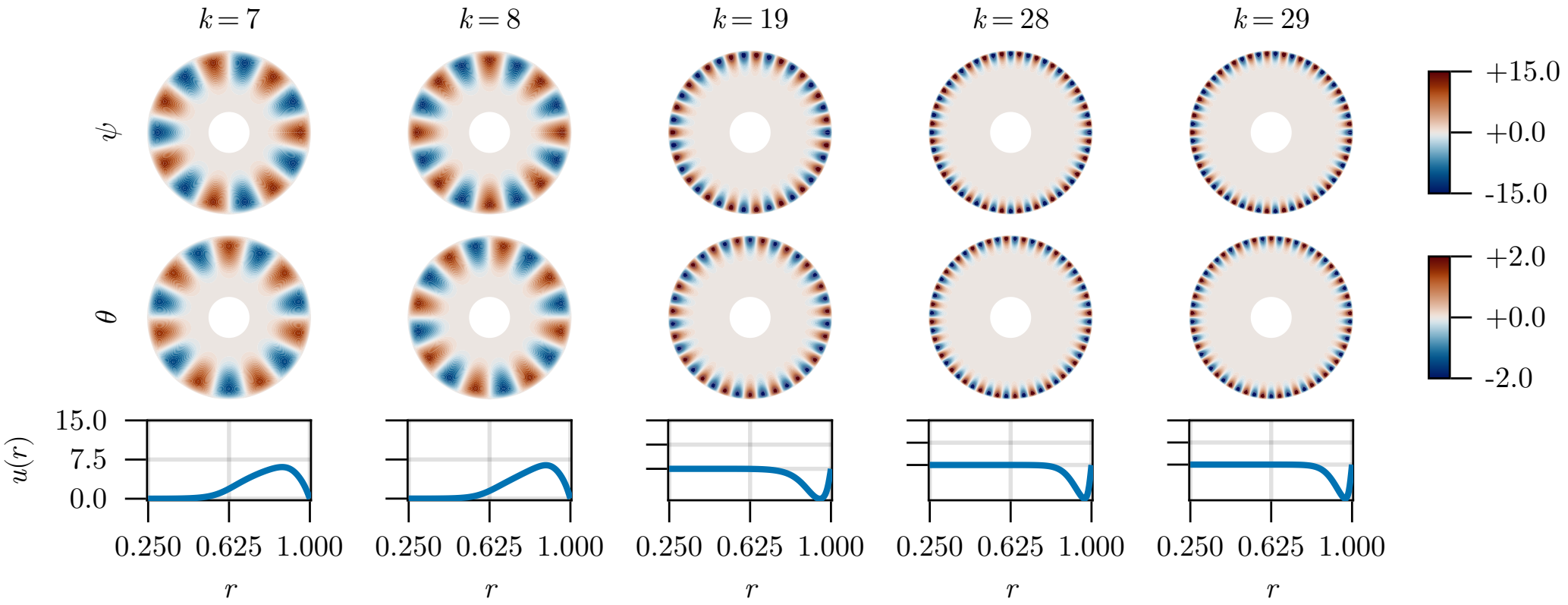


Figure: Critical streamfunction-temperature pairs (ψ, θ) and wavenumbers k for which the constraints are active at $\mathrm{Pe} = 100$. The pairs recovered from functions $u(r)$, also shown, for which the constraints are active.

Annulus example - active wavenumbers

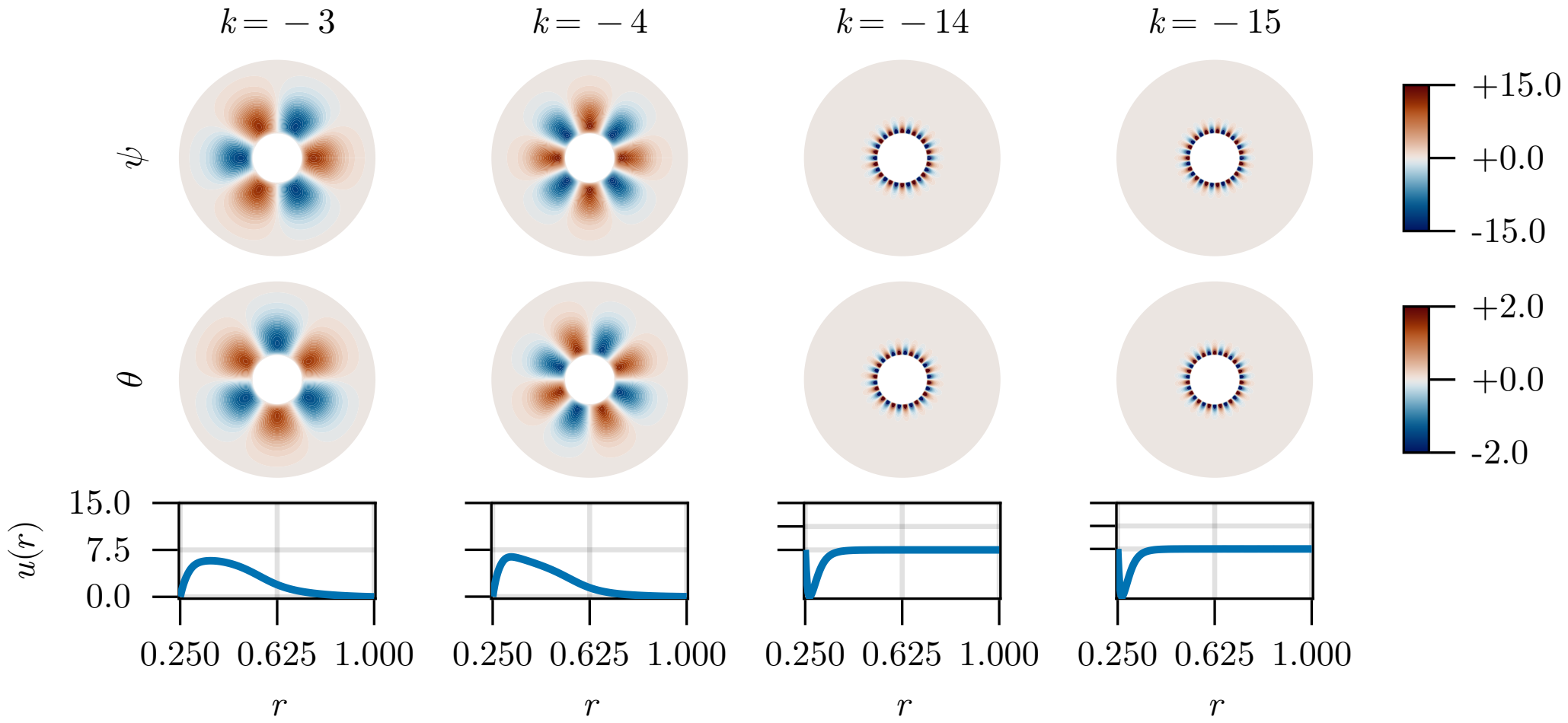


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Improved bounds

Let the azimuthal average of f be $\overline{f}(r) := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(r,\phi)\,\mathrm{d}\phi$.

Theorem 4

Let Ω be a unit annulus or a disk. Assume $\overline{f}(r) \geq f_{\min} > 0$, there exists a constant $C(f_{\min})$ such that

$$\mathcal{J}(\mathrm{Pe}) \geq C\frac{\ln^2 \mathrm{Pe}}{\mathrm{Pe}^2} \qquad \forall\, \mathrm{Pe} > 0.$$



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Main contributions (Braga and Fantuzzi 2026)

1. Formulate lower bounds on the optimal control cost $J(\mathrm{Pe})$ via Lagrange duality, combined with semidefinite programming;
2. Construct feasible points to the dual of (OCP) (∞-dim SDP)
3. Well-posedness of the derived dual
4. Guaranteed convergence by finite-dimensional approximations (Theorem 3)
5. Numerical experiments
6. Analytical upper bounds on $\mathcal{E}(\mathrm{Pe}) \leq 1 + C\mathrm{Pe}^2$, $\forall \mathrm{Pe} \geq 0$
7. Improved bound for disk and annulus $\mathcal{E}(\mathrm{Pe}) \leq C\frac{\mathrm{Pe}^2}{\ln^2 \mathrm{Pe}^2}$, $\forall \mathrm{Pe} > 0$ - inspired by numerics!

Some open questions

- Generic cooling efficiency, different heating and BCs, time-dependance?
- Convergence rates
- Computational bottlenecks
- (OCP) not convex $\Rightarrow$ bounds are not expected to be sharp
- Moment-SOS hierarchies

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Thank you for your attention!
Any questions?